ME 220 Engineering Mechanics - Dynamics
12.1 INTRODUCTION &
12.2 RECTILINEAR KINEMATICS: CONTINUOUS MOTION

Objectives: Students will be able to find the kinematic quantities (position, displacement, velocity, and acceleration) of a particle traveling along a straight path.

- Relations between $s(t)$, $v(t)$, and $a(t)$ for general rectilinear motion.
- Relations between $s(t)$, $v(t)$, and $a(t)$ when acceleration is constant.
APPLICATIONS

The motion of large objects, such as rockets, airplanes, or cars, can often be analyzed as if they were particles.

Why?

If we measure the altitude of this rocket as a function of time, how can we determine its velocity and acceleration?
A sports car travels along a straight road. Can we treat the car as a particle? If the car accelerates at a constant rate, how can we determine its position and velocity at some instant?
An Overview of Mechanics

Mechanics: The study of how bodies react to forces acting on them.

Statics: The study of bodies in equilibrium.

Dynamics:
1. Kinematics – concerned with the geometric aspects of motion
2. Kinetics - concerned with the forces causing the motion
RECTILINEAR KINEMATICS: CONTINUOUS MOTION
(Section 12.2)

A particle travels along a straight-line path defined by the coordinate axis s.

The position of the particle at any instant, relative to the origin, O, is defined by the position vector \( \mathbf{r} \), or the scalar s. Scalar s can be positive or negative. Typical units for \( \mathbf{r} \) and s are meters (m) or feet (ft).

The displacement of the particle is defined as its change in position.

Vector form: \( \Delta \mathbf{r} = \mathbf{r}' - \mathbf{r} \)  
Scalar form: \( \Delta s = s' - s \)

The total distance traveled by the particle, \( s_T \), is a positive scalar that represents the total length of the path over which the particle travels.
VELOCITY

Velocity is a measure of the rate of change in the position of a particle. It is a vector quantity (it has both magnitude and direction). The magnitude of the velocity is called speed, with units of m/s or ft/s.

The average velocity of a particle during a time interval $\Delta t$ is

$$v_{\text{avg}} = \frac{\Delta r}{\Delta t}$$

The instantaneous velocity is the time-derivative of position.

$$v = \frac{dr}{dt}$$

Speed is the magnitude of velocity: $v = \frac{ds}{dt}$

Average speed is the total distance traveled divided by elapsed time:

$$v_{\text{sp}}_{\text{avg}} = \frac{s_T}{\Delta t}$$
ACCELERATION

Acceleration is the rate of change in the velocity of a particle. It is a vector quantity. Typical units are m/s² or ft/s².

The instantaneous acceleration is the time derivative of velocity.

Vector form: \( \mathbf{a} = \frac{d\mathbf{v}}{dt} \)

Scalar form: \( a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \)

Acceleration can be positive (speed increasing) or negative (speed decreasing).

As the book indicates, the derivative equations for velocity and acceleration can be manipulated to get \( a ds = v dv \).
SUMMARY OF KINEMATIC RELATIONS: RECTILINEAR MOTION

• Differentiate position to get velocity and acceleration.

\[ v = \frac{ds}{dt} ; \quad a = \frac{dv}{dt} \quad \text{or} \quad a = v \frac{dv}{ds} \]

• Integrate acceleration for velocity and position.

Velocity:

\[ \int_{v_0}^{v} dv = \int_{t_o}^{t} a \, dt \quad \text{or} \quad \int_{v_0}^{v} v \, dv = \int_{s_o}^{s} a \, ds \]

Position:

\[ \int_{s_o}^{s} ds = \int_{t_o}^{t} v \, dt \]

• Note that \( s_o \) and \( v_o \) represent the initial position and velocity of the particle at \( t = 0 \).
CONSTANT ACCELERATION

The three kinematic equations can be integrated for the special case when acceleration is constant \( (a = a_c) \) to obtain very useful equations. A common example of constant acceleration is gravity; i.e., a body freely falling toward earth. In this case, \( a_c = g = 9.81 \text{ m/s}^2 = 32.2 \text{ ft/s}^2 \) downward. These equations are:

\[
\int_{v_o}^{v} dv = \int_{t_o}^{t} a_c \, dt \quad \text{yields} \quad v = v_o + a_c t
\]

\[
\int_{s_o}^{s} ds = \int_{t_o}^{t} v \, dt \quad \text{yields} \quad s = s_o + v_o t + \frac{1}{2} a_c t^2
\]

\[
\int_{v_o}^{v} v \, dv = \int_{s_o}^{s} a_c \, ds \quad \text{yields} \quad v^2 = (v_o)^2 + 2a_c(s - s_o)
\]
EXAMPLE 12.1

Given: A car moves in a straight line to the right with a velocity of \( v = (0.9 \, t^2 + 0.6 \, t) \) m/s where \( t \) is in seconds. Also, \( s = 0 \) when \( t = 0 \).

Find: The position and acceleration of the particle when \( t = 3 \) s.

Plan: Establish the positive coordinate, \( s \), in the direction the car is traveling. Since the velocity is given as a function of time, take a derivative of it to calculate the acceleration. Conversely, integrate the velocity function to calculate the position.
EXAMPLE

Solution: (continued)

1) Take a derivative of the velocity to determine the acceleration.

\[ a = \frac{dv}{dt} = \frac{d(0.9t^2 + 0.6t)}{dt} = 1.8t + 0.6 \]

=> \[ a = 6 \text{ m/s}^2 \text{ (or in the } \rightarrow \text{ direction) when } t = 3 \text{ s} \]

2) Calculate the distance traveled in 4s by integrating the velocity using \( s_0 = 0 \):

\[ v = \frac{ds}{dt} \Rightarrow ds = v \, dt \Rightarrow \int_{s_0}^{s} ds = \int_{0}^{t} (0.9t^2 + 0.6t) \, dt \]

=> \[ s - s_0 = 0.3t^3 + 0.3t^2 \]

=> \[ s - 0 = 0.3(3)^2 + 0.3(3)^3 \Rightarrow s = 10.8 \text{ m (or } \rightarrow \text{)} \]
EXAMPLE

**Given:** Ball A is released from rest at a height of 40 ft at the same time that ball B is thrown upward, 5 ft from the ground. The balls pass one another at a height of 20 ft.

**Find:** The speed at which ball B was thrown upward.

**Plan:** Both balls experience a constant downward acceleration of 32.2 ft/s² due to gravity. Apply the formulas for constant acceleration, with $a_c = -32.2$ ft/s².
Solution:

1) First consider ball A. With the origin defined at the ground, ball A is released from rest \((v_A)_o = 0\) at a height of 40 ft \((s_A)_o = 40 \text{ ft}\). Calculate the time required for ball A to drop to 20 ft \((s_A = 20 \text{ ft})\) using a position equation.

\[
s_A = (s_A)_o + (v_A)_o t + \frac{1}{2} a_c t^2
\]

So,

\[
20 \text{ ft} = 40 \text{ ft} + (0)(t) + \frac{1}{2}(-32.2)(t^2) \quad \Rightarrow \quad t = 1.115 \text{ s}
\]
EXAMPLE
(continued)

Solution:

2) Now consider ball B. It is throw upward from a height of 5 ft \(((s_B)_o = 5 \text{ ft})\). It must reach a height of 20 ft \((s_B = 20 \text{ ft})\) at the same time ball A reaches this height \((t = 1.115 \text{ s})\). Apply the position equation again to ball B using \(t = 1.115s\).

\[
s_B = (s_B)_o + (v_B)_o t + \frac{1}{2} a_c t^2
\]

So,

\[
20 \text{ ft} = 5 + (v_B)_o(1.115) + \frac{1}{2}(-32.2)(1.115)^2
\]

\[
=> (v_B)_o = 31.4 \text{ ft/s}
\]
EXAMPLE 12.2
During a test a rocket travels upwards at 75 m/s, and when it is 40 m from the ground its engine fails. Determine the maximum height \( s_B \) reached by the rocket and its speed just before it hits the ground. While in motion the rocket is subjected to a constant downward acceleration of 9.81 m/s\(^2\) due to gravity. Neglect the effect of air resistance.
EXAMPLE 12.2 (continued)
12.3 Rectilinear Kinematics: Erratic Motion

Objectives:
Students will be able to determine position, velocity, and acceleration of a particle using graphs.

$s = 0.4t^2$

$s = 24t - 360$

$s - t$, $v - t$, $a - t$, $v - s$, and $a - s$ diagrams
In many experiments, a velocity versus position (v-s) profile is obtained.

If we have a v-s graph for the tank truck, how can we determine its acceleration at position \( s = 1500 \) feet?
Graphing provides a good way to handle complex motions that would be difficult to describe with formulas.

Graphs also provide a visual description of motion and reinforce the calculus concepts of differentiation and integration as used in dynamics.

The approach builds on the facts that slope and differentiation are linked and that integration can be thought of as finding the area under a curve.
S-T GRAPH

Plots of position vs. time can be used to find velocity vs. time curves. Finding the slope of the line tangent to the motion curve at any point is the velocity at that point (or $v = \frac{ds}{dt}$).

Therefore, the v-t graph can be constructed by finding the slope at various points along the s-t graph.
V-T GRAPH

Plots of velocity vs. time can be used to find acceleration vs. time curves. Finding the slope of the line tangent to the velocity curve at any point is the acceleration at that point (or \( a = \frac{dv}{dt} \)).

Therefore, the acceleration vs. time (or a-t) graph can be constructed by finding the slope at various points along the v-t graph.

Also, the distance moved (displacement) of the particle is the area under the v-t graph during time \( \Delta t \).
A-T GRAPH

Given the acceleration vs. time or a-t curve, the change in velocity ($\Delta v$) during a time period is the area under the a-t curve.

So we can construct a v-t graph from an a-t graph if we know the initial velocity of the particle.
A more complex case is presented by the acceleration versus position or a-s graph. The area under the a-s curve represents the change in velocity (recall $\int a \, ds = \int v \, dv$).

$$\frac{1}{2} (v_1^2 - v_0^2) = \int_{s_1}^{s_2} a \, ds = \text{area under the a-s graph}$$

This equation can be solved for $v_1$, allowing you to solve for the velocity at a point. By doing this repeatedly, you can create a plot of velocity versus distance.
V-S GRAPH

Another complex case is presented by the velocity vs. distance or v-s graph. By reading the velocity \( v \) at a point on the curve and multiplying it by the slope of the curve (\( \frac{dv}{ds} \)) at this same point, we can obtain the acceleration at that point. Recall the formula

\[
a = v \left( \frac{dv}{ds} \right).
\]

Thus, we can obtain an a-s plot from the v-s curve.
EXAMPLE from F12-12

Given: The s-t graph for a sports car moving along a straight road.

Find: The v-t graph and a-t graph over the time interval

\[0 \leq t \leq 10s\]
Solution: The v-t graph can be constructed by finding the slope of the s-t graph at key points. What are those?

when $0 < t < 5$ s; $v_{0-5} = \frac{ds}{dt} = \frac{d(3t^2)}{dt} = 6t$ m/s

when $5 < t < 10$ s; $v_{5-10} = \frac{ds}{dt} = \frac{d(30t-75)}{dt} = 30$ m/s
EXAMPLE (continued)

Similarly, the a-t graph can be constructed by finding the slope at various points along the v-t graph.

when $0 < t < 5$ s; \[ a_{0-5} = \frac{dv}{dt} = \frac{d(6t)}{dt} = 6 \text{ m/s}^2 \]

when $5 < t < 10$ s; \[ a_{5-10} = \frac{dv}{dt} = \frac{d(30)}{dt} = 0 \text{ m/s}^2 \]
EXAMPLE from P12-51

Given: The v-t graph shown.

Find: The a-t graph, average speed, and distance traveled for the 0 - 90 s interval.

Plan:

Find slopes of the v-t curve and draw the a-t graph.
Find the area under the curve. It is the distance traveled.
Finally, calculate average speed (using basic definitions!).
EXAMPLE
(continued)

Solution:

Find the a–t graph:

For $0 \leq t \leq 30$  \( a = \frac{dv}{dt} = 1.0 \ \text{m/s}^2 \)

For $30 \leq t \leq 90$  \( a = \frac{dv}{dt} = -0.5 \ \text{m/s}^2 \)
EXAMPLE (continued)

Now find the distance traveled:

$$\Delta s_{0-30} = \int v \, dt = \frac{1}{2} (30)^2 = 450 \text{ m}$$

$$\Delta s_{30-90} = \int v \, dt$$

$$= \frac{1}{2} (-0.5)(90)^2 + 45(90) - \frac{1}{2} (-0.5)(30)^2 - 45(30)$$

$$= 900 \text{ m}$$

$$s_{0-90} = 450 + 900 = 1350 \text{ m}$$

$$v_{\text{avg}(0-90)} = \text{total distance} / \text{time}$$

$$= 1350 / 90$$

$$= 15 \text{ m/s}$$
EXAMPLE 12.8
The v-s graph describing the motion of a motorcycle is shown in Fig.12-15a. Construct the a-s graph of the motion and determine the time needed for the motorcycle to reach the position $s = 120$ m.
EXAMPLE 12.8 (continued)
End of the Lecture

Let Learning Continue
Objectives:
Students will be able to:
1. Describe the motion of a particle traveling along a curved path.
2. Relate kinematic quantities in terms of the rectangular components of the vectors.

- General Curvilinear Motion
- Rectangular Components of Kinematic Vectors
**APPLICATIONS**

The path of motion of a plane can be tracked with radar and its x, y, and z coordinates (relative to a point on earth) recorded as a function of time.

How can we determine the velocity or acceleration of the plane at any instant?
A roller coaster car travels down a fixed, helical path at a constant speed.

How can we determine its position or acceleration at any instant?

If you are designing the track, why is it important to be able to predict the acceleration of the car?
GENERAL CURVILINEAR MOTION
(Section 12.4)

A particle moving along a curved path undergoes curvilinear motion. Since the motion is often three-dimensional, vectors are used to describe the motion.

A particle moves along a curve defined by the path function, s.

The position of the particle at any instant is designated by the vector \( r = r(t) \). Both the magnitude and direction of \( r \) may vary with time.

If the particle moves a distance \( \Delta s \) along the curve during time interval \( \Delta t \), the displacement is determined by vector subtraction: \( \Delta r = r' - r \)
Velocity represents the rate of change in the position of a particle.

The average velocity of the particle during the time increment $\Delta t$ is

$$v_{\text{avg}} = \frac{\Delta r}{\Delta t}.$$ 

The instantaneous velocity is the time-derivative of position

$$v = \frac{dr}{dt}.$$ 

The velocity vector, $v$, is always tangent to the path of motion.

The magnitude of $v$ is called the speed. Since the arc length $\Delta s$ approaches the magnitude of $\Delta r$ as $t \to 0$, the speed can be obtained by differentiating the path function ($v = ds/dt$). Note that this is not a vector!
ACCELERATION

Acceleration represents the rate of change in the velocity of a particle.

If a particle’s velocity changes from \( v \) to \( v' \) over a time increment \( \Delta t \), the average acceleration during that increment is:

\[
a_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{(v - v')}{\Delta t}
\]

The instantaneous acceleration is the time-derivative of velocity:

\[
a = \frac{dv}{dt} = \frac{d^2r}{dt^2}
\]

A plot of the locus of points defined by the arrowhead of the velocity vector is called a hodograph. The acceleration vector is tangent to the hodograph, but not, in general, tangent to the path function.
It is often convenient to describe the motion of a particle in terms of its x, y, z or rectangular components, relative to a fixed frame of reference.

The position of the particle can be defined at any instant by the position vector

\[ \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \, . \]

The x, y, z components may all be functions of time, i.e.,

\[ x = x(t), \quad y = y(t), \quad \text{and} \quad z = z(t) \, . \]

The magnitude of the position vector is:

\[ r = \left(x^2 + y^2 + z^2\right)^{0.5} \]

The direction of \( \mathbf{r} \) is defined by the unit vector:

\[ \mathbf{u}_r = \left(1/r\right)\mathbf{r} \]
RECTANGULAR COMPONENTS: VELOCITY

The velocity vector is the time derivative of the position vector:

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d(x\mathbf{i})}{dt} + \frac{d(y\mathbf{j})}{dt} + \frac{d(z\mathbf{k})}{dt} \]

Since the unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are constant in magnitude and direction, this equation reduces to

\[ \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \]

where \( v_x = \dot{x} = \frac{dx}{dt}, \quad v_y = \dot{y} = \frac{dy}{dt}, \quad v_z = \dot{z} = \frac{dz}{dt} \)

The magnitude of the velocity vector is

\[ v = \left[ (v_x)^2 + (v_y)^2 + (v_z)^2 \right]^{0.5} \]

The direction of \( \mathbf{v} \) is tangent to the path of motion.
RECTANGULAR COMPONENTS: ACCELERATION

The acceleration vector is the time derivative of the velocity vector (second derivative of the position vector):

\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \]

where \( a_x = \ddot{x} = \mathbf{\ddot{x}} = \frac{dv_x}{dt}, \quad a_y = \ddot{y} = \mathbf{\ddot{y}} = \frac{dv_y}{dt}, \quad a_z = \ddot{z} = \mathbf{\ddot{z}} = \frac{dv_z}{dt} \)

The magnitude of the acceleration vector is

\[ a = \left[ (a_x)^2 + (a_y)^2 + (a_z)^2 \right]^{0.5} \]

The direction of \( \mathbf{a} \) is usually not tangent to the path of the particle.
EXAMPLE I

**Given:** The motion of two particles (A and B) is described by the position vectors

\[ r_A = [3t \, i + 9t(2 - t) \, j] \, \text{m} \quad \text{and} \quad r_B = [3(t^2 - 2t + 2) \, i + 3(t - 2) \, j] \, \text{m}. \]

**Find:** The point at which the particles collide and their speeds just before the collision.

![Diagram showing particles A and B at positions A and B with vector displacements](diagram.png)
EXAMPLE I (continued)

Solution:

1) The point of collision requires that $\mathbf{r}_A = \mathbf{r}_B$, 
so $x_A = x_B$ and $y_A = y_B$.

Set the x-components equal: $3t = 3(t^2 - 2t + 2)$
   Simplifying: $t^2 - 3t + 2 = 0$
   Solving: $t = \{3 \pm [3^2 - 4(1)(2)]^{0.5}\}/2(1)$
   $\Rightarrow t = 2$ or $1$ s

Set the y-components equal: $9t(2 - t) = 3(t - 2)$
   Simplifying: $3t^2 - 5t - 2 = 0$
   Solving: $t = \{5 \pm [5^2 - 4(3)(-2)]^{0.5}\}/2(3)$
   $\Rightarrow t = 2$ or $-1/3$ s

So, the particles collide when $t = 2$ s (only common time). 
Substituting this value into $\mathbf{r}_A$ or $\mathbf{r}_B$ yields

$x_A = x_B = 6$ m \ and \ $y_A = y_B = 0$
EXAMPLE I (continued)

2) Differentiate $r_A$ and $r_B$ to get the velocity vectors.

$$
\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = \dot{x}_A \mathbf{i} + \dot{y}_A \mathbf{j} = \left[ 3 \mathbf{i} + (18 - 18t) \mathbf{j} \right] \text{m/s}
$$

At $t = 2$ s: $\mathbf{v}_A = \left[ 3\mathbf{i} - 18 \mathbf{j} \right] \text{m/s}$

$$
\mathbf{v}_B = \frac{d\mathbf{r}_B}{dt} = x_B \mathbf{i} + y_B \mathbf{j} = \left[ (6t - 6) \mathbf{i} + 3 \mathbf{j} \right] \text{m/s}
$$

At $t = 2$ s: $\mathbf{v}_B = \left[ 6 \mathbf{i} + 3 \mathbf{j} \right] \text{m/s}$

Speed is the magnitude of the velocity vector.

$$
\mathbf{v}_A = (3^2 + 18^2)^{0.5} = 18.2 \text{ m/s}
$$

$$
\mathbf{v}_B = (6^2 + 3^2)^{0.5} = 6.71 \text{ m/s}
$$
EXAMPLE II (P12-74)

Given: The velocity of the particle is

\[ v = [16 t^2 i + 4 t^3 j + (5 t + 2) k] \text{ m/s.} \]

When \( t = 0 \), \( x = y = z = 0 \).

Find: The particle’s coordinate position and the magnitude of its acceleration when \( t = 2 \text{ s} \).

Plan:

Note that velocity vector is given as a function of time.

1) Determine the position and acceleration by integrating and differentiating \( v \), respectively, using the initial conditions.

2) Determine the magnitude of the acceleration vector using \( t = 2 \text{ s} \).
EXAMPLE II (P12-74) (continued)

Solution:

1) x-components:
Velocity known as: \( v_x = \dot{x} = dx/dt = (16 \ t^2) \) m/s

\[
\int_x^t dx = \int_0^t (16 \ t^2) \ dt \Rightarrow x = (16/3)t^3 = 42.7 \text{ m at } t = 2 \text{ s}
\]

Acceleration: \( a_x = \ddot{x} = \dot{v}_x = d/dt (16 \ t^2) = 32 \ t = 64 \text{ m/s}^2 \)

2) y-components:
Velocity known as: \( v_y = \dot{y} = dy/dt = (4 \ t^3) \) m/s

\[
\int_y^t dy = \int_0^t (4 \ t^3) \ dt \Rightarrow y = t^4 = (16) \text{ m at } t = 2 \text{ s}
\]

Acceleration: \( a_y = \ddot{y} = \dot{v}_y = d/dt (4 \ t^3) = 12 \ t^2 = 48 \text{ m/s}^2 \)
3) z-components:

Velocity is known as: \( v_z = \dot{z} = \frac{dz}{dt} = (5 \ t + 2) \ m/s \)

Position: \( \int_0^t \dot{z} \, dz = \int_0^t (5 \ t + 2) \, dt \Rightarrow z = (5/2) \ t^2 + 2t = 14 \ m \) at \( t=2s \)

Acceleration: \( a_z = \ddot{z} = \dot{v}_z = \frac{d}{dt} (5 \ t + 2) = 5 \ m/s^2 \)

4) The position vector and magnitude of the acceleration vector are written using the component information found above.

Position vector: \( \mathbf{r} = [42.7 \ \mathbf{i} + 16 \ \mathbf{j} + 14 \ \mathbf{k}] \ m. \)

Acceleration vector: \( \mathbf{a} = [64 \ \mathbf{i} + 48 \ \mathbf{j} + 5 \ \mathbf{k}] \ m/s^2 \)

Magnitude: \( a = (64^2 + 48^2 + 5^2)^{0.5} = 80.2 \ m/s^2 \)
EXAMPLE 12.9 At any instant the horizontal position of the weather balloon in Fig 12-18a is defined by $x = (9t)$ m, where $t$ is in seconds. If the equation of the path is $y = \frac{x^2}{30}$, determine the magnitude and direction of the velocity and the acceleration when $t = 2$ s.
EXAMPLE 12.9 (continued)
Objectives:
Students will be able to analyze the free-flight motion of a projectile.

• Kinematic Equations for Projectile Motion
A good kicker instinctively knows at what angle, \( \theta \), and initial velocity, \( v_A \), he must kick the ball to make a field goal.

For a given kick “strength”, at what angle should the ball be kicked to get the maximum distance?
A basketball is shot at a certain angle. What parameters should the shooter consider in order for the basketball to pass through the basket?

Distance, speed, the basket location, … anything else?
A firefighter needs to know the maximum height on the wall she can project water from the hose. What parameters would you program into a wrist computer to find the angle, $\theta$, that she should use to hold the hose?
Projectile motion can be treated as two rectilinear motions, one in the horizontal direction experiencing zero acceleration and the other in the vertical direction experiencing constant acceleration (i.e., from gravity).

For illustration, consider the two balls on the left. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity. Each picture in this sequence is taken after the same time interval. Notice both balls are subjected to the same downward acceleration since they remain at the same elevation at any instant. Also, note that the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction is constant.
Since \( a_x = 0 \), the velocity in the horizontal direction remains constant (\( v_x = v_{ox} \)) and the position in the x direction can be determined by:

\[
x = x_0 + (v_{ox}) \, t
\]

Why is \( a_x \) equal to zero (assuming movement through the air)?
Since the positive y-axis is directed upward, $a_y = -g$. Application of the constant acceleration equations yields:

$$v_y = v_{oy} - g t$$

$$y = y_o + (v_{oy}) t - \frac{1}{2} g t^2$$

$$v_y^2 = v_{oy}^2 - 2 g (y - y_o)$$

For any given problem, only two of these three equations can be used. Why?
EXAMPLE 1

Given: $v_o$ and $\theta$

Find: The equation that defines $y$ as a function of $x$.

Plan: Eliminate time from the kinematic equations.

Solution: Using $v_x = v_o \cos \theta$ and $v_y = v_o \sin \theta$

We can write: $x = (v_o \cos \theta)t$ or $t = \frac{x}{v_o \cos \theta}$

$y = (v_o \sin \theta) t - \frac{1}{2} g (t)^2$

By substituting for $t$:

$y = (v_o \sin \theta) \left\{ \frac{x}{v_o \cos \theta} \right\}^2 - \left\{ \frac{g}{v_o \cos \theta} \right\} x$
EXAMPLE I (continued)

Simplifying the last equation, we get:

\[ y = (x \tan \theta) - \frac{g x^2}{2v_0^2} (1 + \tan^2 \theta) \]

The above equation is called the “path equation” which describes the path of a particle in projectile motion. The equation shows that the path is parabolic.
EXAMPLE II

**Given:** Projectile is fired with $v_A=150$ m/s at point A.

**Find:** The horizontal distance it travels (R) and the time in the air.

**Plan:**

Establish a fixed x, y coordinate system (in this solution, the origin of the coordinate system is placed at A). Apply the kinematic relations in x- and y-directions.
EXAMPLE II (continued)

Solution:
1) Place the coordinate system at point A.
   Then, write the equation for horizontal motion.
   \[ \rightarrow \quad x_B = x_A + v_{Ax} t_{AB} \]
   where \( x_B = R \), \( x_A = 0 \), \( v_{Ax} = 150 \, (4/5) \, \text{m/s} \)

   Range, \( R \) will be \( R = 120 \, t_{AB} \)

2) Now write a vertical motion equation. Use the distance equation.
   \[ \uparrow + \quad y_B = y_A + v_{Ay} t_{AB} - 0.5 \, g \, t_{AB}^2 \]
   where \( y_B = -150 \), \( y_A = 0 \), and \( v_{Ay} = 150(3/5) \, \text{m/s} \)

   We get the following equation: \(-150 = 90 \, t_{AB} + 0.5 \, (-9.81) \, t_{AB}^2\)

   Solving for \( t_{AB} \) first, \( t_{AB} = 19.89 \, \text{s} \).
   Then, \( R = 120 \, t_{AB} = 120 \, (19.89) = 2387 \, \text{m} \)
End of the Lecture

Let Learning Continue
12-7 CURVILINEAR MOTION:
NORMAL AND TANGENTIAL COMPONENTS

Objectives:
Students will be able to determine the normal and tangential components of velocity and acceleration of a particle traveling along a curved path.

• Normal and Tangential Components of Velocity and Acceleration
• Special Cases of Motion
APPLICATIONS

Cars traveling along a clover-leaf interchange experience an acceleration due to a change in velocity as well as due to a change in direction of the velocity.

If the car’s speed is increasing at a known rate as it travels along a curve, how can we determine the magnitude and direction of its total acceleration?

Why would you care about the total acceleration of the car?
A roller coaster travels down a hill for which the path can be approximated by a function $y = f(x)$.

The roller coaster starts from rest and increases its speed at a constant rate.

How can we determine its velocity and acceleration at the bottom?

Why would we want to know these values?
NORMAL AND TANGENTIAL COMPONENTS
(Section 12.7)

When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal (n) and tangential (t) coordinates are often used.

In the n-t coordinate system, the origin is located on the particle (the origin moves with the particle).

The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle’s motion. The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.
The positive n and t directions are defined by the unit vectors $u_n$ and $u_t$, respectively.

The center of curvature, O’, always lies on the concave side of the curve. The radius of curvature, $\rho$, is defined as the perpendicular distance from the curve to the center of curvature at that point.

The position of the particle at any instant is defined by the distance, $s$, along the curve from a fixed reference point.
VELOCITY IN THE n-t COORDINATE SYSTEM

The velocity vector is always tangent to the path of motion (t-direction).

The magnitude is determined by taking the time derivative of the path function, \( s(t) \).

\[
v = v \, u_t \quad \text{where} \quad v = \dot{s} = \frac{ds}{dt}
\]

Here, \( v \) defines the magnitude of the velocity (speed) and \( u_t \) defines the direction of the velocity vector.
ACCELERATION IN THE $n$-$t$ COORDINATE SYSTEM

Acceleration is the time rate of change of velocity:

$$a = \frac{dv}{dt} = d(\nu u_t)/dt = \dot{\nu} u_t + \nu \dot{u}_t$$

Here $\dot{\nu}$ represents the change in the magnitude of velocity and $\dot{u}_t$ represents the rate of change in the direction of $u_t$.

After mathematical manipulation, the acceleration vector can be expressed as:

$$a = \dot{\nu} u_t + (\nu^2/\rho) u_n = a_t u_t + a_n u_n.$$
ACCELERATION IN THE n-t COORDINATE SYSTEM
(continued)

So, there are two components to the acceleration vector:
\[ \mathbf{a} = a_t \mathbf{u}_t + a_n \mathbf{u}_n \]

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.
  \[ a_t = \dot{\mathbf{v}} \quad \text{or} \quad a_t \, ds = \mathbf{v} \, dv \]
- The normal or centripetal component is always directed toward the center of curvature of the curve.
  \[ a_n = \mathbf{v}^2 / \rho \]
- The magnitude of the acceleration vector is
  \[ a = [(a_t)^2 + (a_n)^2]^{0.5} \]
SPECIAL CASES OF MOTION

There are some special cases of motion to consider.

1) The particle moves along a straight line.
   \[ \rho \to \infty \Rightarrow a_n = \frac{v^2}{\rho} = 0 \Rightarrow a = a_t = \dot{v} \]
   The tangential component represents the time rate of change in the magnitude of the velocity.

2) The particle moves along a curve at constant speed.
   \[ a_t = \dot{v} = 0 \Rightarrow a = a_n = \frac{v^2}{\rho} \]
   The normal component represents the time rate of change in the direction of the velocity.
3) The tangential component of acceleration is constant, $a_t = (a_t)_c$. In this case,

$$s = s_o + v_o t + (1/2) (a_t)_c t^2$$

$$v = v_o + (a_t)_c t$$

$$v^2 = (v_o)^2 + 2 (a_t)_c (s - s_o)$$

As before, $s_o$ and $v_o$ are the initial position and velocity of the particle at $t = 0$. How are these equations related to projectile motion equations? Why?

4) The particle moves along a path expressed as $y = f(x)$. The radius of curvature, $\rho$, at any point on the path can be calculated from

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$
THREE-DIMENSIONAL MOTION

If a particle moves along a space curve, the n and t axes are defined as before. At any point, the t-axis is tangent to the path and the n-axis points toward the center of curvature. The plane containing the n and t axes is called the osculating plane.

A third axis can be defined, called the binomial axis, b. The binomial unit vector, \( \mathbf{u}_b \), is directed perpendicular to the osculating plane, and its sense is defined by the cross product \( \mathbf{u}_b = \mathbf{u}_t \times \mathbf{u}_n \).

There is no motion, thus no velocity or acceleration, in the binomial direction.
EXAMPLE I

Given: A boat travels around a circular path, $\rho = 40$ m, at a speed that increases with time, $v = (0.0625 t^2)$ m/s.

Find: The magnitudes of the boat’s velocity and acceleration at the instant $t = 10$ s.

Plan:

The boat starts from rest ($v = 0$ when $t = 0$).

1) Calculate the velocity at $t = 10$ s using $v(t)$.
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.
EXAMPLE (continued)

Solution:

1) The velocity vector is \( \mathbf{v} = \mathbf{v} \mathbf{u}_t \), where the magnitude is given by \( \mathbf{v} = (0.0625t^2) \) m/s. At \( t = 10s \):

\[
\mathbf{v} = 0.0625 \, t^2 = 0.0625 \, (10)^2 = 6.25 \, \text{m/s}
\]

2) The acceleration vector is \( \mathbf{a} = a_t \mathbf{u}_t + a_n \mathbf{u}_n = \dot{\mathbf{v}} \mathbf{u}_t + \left( \frac{\mathbf{v}^2}{\rho} \right) \mathbf{u}_n \).

Tangential component: \( a_t = \dot{\mathbf{v}} = d(0.0625 \, t^2) / dt = 0.125 \, t \, \text{m/s}^2 \)

At \( t = 10s \): \( a_t = 0.125t = 0.125(10) = 1.25 \, \text{m/s}^2 \)

Normal component: \( a_n = \frac{\mathbf{v}^2}{\rho} \, \text{m/s}^2 \)

At \( t = 10s \): \( a_n = (6.25)^2 / (40) = 0.9766 \, \text{m/s}^2 \)

The magnitude of the acceleration is

\[
a = [(a_t)^2 + (a_n)^2]^{0.5} = [(1.25)^2 + (0.9766)^2]^{0.5} = 1.59 \, \text{m/s}^2
\]
EXAMPLE II

Given: A roller coaster travels along a vertical parabolic path defined by the equation \( y = 0.01x^2 \). At point B, it has a speed of 25 m/s, which is increasing at the rate of 3 m/s\(^2\).

Find: The magnitude of the roller coaster’s acceleration when it is at point B.

Plan:

1. The change in the speed of the car (3 m/s\(^2\)) is the tangential component of the total acceleration.
2. Calculate the radius of curvature of the path at B.
3. Calculate the normal component of acceleration.
4. Determine the magnitude of the acceleration vector.
EXAMPLE II

Solution:

1) The tangential component of acceleration is the rate of increase of the roller coaster’s speed, so \( a_t = \ddot{v} = 3 \text{ m/s}^2 \).

2) Determine the radius of curvature at point B \((x = 30 \text{ m})\):

\[
\frac{dy}{dx} = \frac{d(0.01x^2)}{dx} = 0.02x, \quad \frac{d^2y}{dx^2} = \frac{d(0.02x)}{dx} = 0.02
\]

At \(x = 30\) m, \(\frac{dy}{dx} = 0.02(30) = 0.6, \frac{d^2y}{dx^2} = 0.02\)

\[
\rho = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}} = [1 + (0.6)^2]^{3/2}/(0.02) = 79.3 \text{ m}
\]

3) The normal component of acceleration is

\[ a_n = \frac{v^2}{\rho} = \frac{(25)^2}{(79.3)} = 7.881 \text{ m/s}^2 \]

4) The magnitude of the acceleration vector is

\[ a = [(a_t)^2 + (a_n)^2]^{0.5} = [(3)^2 + (7.881)^2]^{0.5} = 8.43 \text{ m/s}^2 \]
12-8 CURVILINEAR MOTION: CYLINDRICAL COMPONENTS

Objectives:
Students will be able to:
1. Determine velocity and acceleration components using cylindrical coordinates.

- Velocity Components
- Acceleration Components
The cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

In the figure shown, the box slides down the helical ramp. How would you find the box’s velocity components to know if the package will fly off the ramp?
We can express the location of P in polar coordinates as \( r = r \, u_r \). Note that the radial direction, \( r \), extends outward from the fixed origin, O, and the transverse coordinate, \( \theta \), is measured counterclockwise (CCW) from the horizontal.
VELOCITY in POLAR COORDINATES)

The instantaneous velocity is defined as:
\[ v = \frac{dr}{dt} = \frac{d(ru_r)}{dt} \]
\[ v = iu_r + r \frac{du_r}{dt} \]

Using the chain rule:
\[ \frac{du_r}{dt} = (\frac{du_r}{d\theta})(\frac{d\theta}{dt}) \]
We can prove that \( \frac{du_r}{d\theta} = u_\theta \) so \( du_r/dt = \dot{\theta}u_\theta \)
Therefore: \[ v = iu_r + r \dot{\theta}u_\theta \]

Thus, the velocity vector has two components: \( i \), called the radial component, and \( r \dot{\theta} \) called the transverse component. The speed of the particle at any given instant is the sum of the squares of both components or
\[ v = \sqrt{(r \dot{\theta})^2 + (i)^2} \]
ACCELERATION (POLAR COORDINATES)

The instantaneous acceleration is defined as:

$$a = \frac{dv}{dt} = (d/dt)(\dot{r}u_r + r\dot{\theta}u_\theta)$$

After manipulation, the acceleration can be expressed as

$$a = (\ddot{r} - r\dot{\theta}^2)u_r + (\ddot{\theta} + 2\dot{r}\dot{\theta})u_\theta$$

The term $(\ddot{r} - r\dot{\theta}^2)$ is the radial acceleration or $a_r$.

The term $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the transverse acceleration or $a_\theta$.

The magnitude of acceleration is

$$a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$$
CYLINDRICAL COORDINATES

If the particle P moves along a space curve, its position can be written as

\[ r_P = ru_r + zu_z \]

Taking time derivatives and using the chain rule:

Velocity: \[ \mathbf{v}_P = \dot{r} \mathbf{u}_r + r\dot{\theta} \mathbf{u}_\theta + \dot{z} \mathbf{u}_z \]

Acceleration: \[ \mathbf{a}_P = (\ddot{r} - r\ddot{\theta}^2) \mathbf{u}_r + (r\dddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{u}_\theta + \ddot{z} \mathbf{u}_z \]
EXAMPLE I

Given: A car travels along a circular path.
\[ r = 300 \text{ ft}, \; \dot{\theta} = 0.4 \text{ (rad/s)}, \]
\[ \ddot{\theta} = 0.2 \text{ (rad/s}^2) \]

Find: Velocity and acceleration

Plan: Use the polar coordinate system.

Solution:
\[ r = 300 \text{ ft}, \; \dot{r} = \ddot{r} = 0, \; \text{and} \; \dot{\theta} = 0.4 \text{ (rad/s)}, \; \ddot{\theta} = 0.2 \text{ (rad/s}^2) \]

Substitute in the equation for velocity
\[ \mathbf{v} = \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta = 0 \mathbf{u}_r + 300 (0.4) \mathbf{u}_\theta \]
\[ \mathbf{v} = \sqrt{(0)^2 + (120)^2} = 120 \text{ ft/s} \]
Substitute in the equation for acceleration:

\[ a = (\ddot{r} - r\dot{\theta}^2)u_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})u_\theta \]

\[ a = [0 - 300(0.4)^2] u_r + [300(0.2) + 2(0)(0.4)] u_\theta \]

\[ a = -48 u_r + 60 u_\theta \text{ ft/s}^2 \]

\[ a = \sqrt{(-48)^2 + (60)^2} = 76.8 \text{ ft/s}^2 \]
12 - 9 ABSOLUTE DEPENDENT MOTION ANALYSIS OF TWO PARTICLES

Objectives:
Students will be able to relate the positions, velocities, and accelerations of particles undergoing dependent motion.

- Define Dependent Motion
- Develop Position, Velocity, and Acceleration Relationships
APPLICATIONS

The cable and pulley system shown can be used to modify the speed of the mine car, A, relative to the speed of the motor, M.

It is important to establish the relationships between the various motions in order to determine the power requirements for the motor and the tension in the cable.

For instance, if the speed of the cable (P) is known because we know the motor characteristics, how can we determine the speed of the mine car? Will the slope of the track have any impact on the answer?
Rope and pulley arrangements are often used to assist in lifting heavy objects. The total lifting force required from the truck depends on both the weight and the acceleration of the cabinet.

How can we determine the acceleration and velocity of the cabinet if the acceleration of the truck is known?
DEPENDENT MOTION (Section 12.9)

In many kinematics problems, the motion of one object will depend on the motion of another object.

The blocks in this figure are connected by an inextensible cord wrapped around a pulley. If block A moves downward along the inclined plane, block B will move up the other incline.

The motion of each block can be related mathematically by defining position coordinates, \(s_A\) and \(s_B\). Each coordinate axis is defined from a fixed point or datum line, measured positive along each plane in the direction of motion of each block.
In this example, position coordinates $s_A$ and $s_B$ can be defined from fixed datum lines extending from the center of the pulley along each incline to blocks A and B.

If the cord has a fixed length, the position coordinates $s_A$ and $s_B$ are related mathematically by the equation

$$s_A + l_{CD} + s_B = l_T$$

Here $l_T$ is the total cord length and $l_{CD}$ is the length of cord passing over the arc CD on the pulley.
The velocities of blocks A and B can be related by differentiating the position equation. Note that $l_{CD}$ and $l_T$ remain constant, so $dl_{CD}/dt = dl_T/dt = 0$

$$\frac{ds_A}{dt} + \frac{ds_B}{dt} = 0 \quad \Rightarrow \quad v_B = -v_A$$

The negative sign indicates that as A moves down the incline (positive $s_A$ direction), B moves up the incline (negative $s_B$ direction).

Accelerations can be found by differentiating the velocity expression. Prove to yourself that $a_B = -a_A$. 

DEPENDENT MOTION EXAMPLE

Consider a more complicated example. Position coordinates \((s_A\) and \(s_B)\) are defined from fixed datum lines, measured along the direction of motion of each block.

Note that \(s_B\) is only defined to the center of the pulley above block B, since this block moves with the pulley. Also, \(h\) is a constant.

The red colored segments of the cord remain constant in length during motion of the blocks.
The position coordinates are related by the equation
\[ 2s_B + h + s_A = l_T \]
Where \( l_T \) is the total cord length minus the lengths of the red segments.

Since \( l_T \) and \( h \) remain constant during the motion, the velocities and accelerations can be related by two successive time derivatives:
\[ 2v_B = -v_A \quad \text{and} \quad 2a_B = -a_A \]

When block B moves downward (+\( s_B \)), block A moves to the left (-\( s_A \)). Remember to be consistent with your sign convention!
DEPENDENT MOTION EXAMPLE (continued)

This example can also be worked by defining the position coordinate for B \((s_B)\) from the bottom pulley instead of the top pulley.

The position, velocity, and acceleration relations then become

\[ 2(h - s_B) + h + s_A = l_T \]
and
\[ 2v_B = v_A \quad 2a_B = a_A \]

Prove to yourself that the results are the same, even if the sign conventions are different than the previous formulation.
DEPENDENT MOTION: PROCEDURES

These procedures can be used to relate the dependent motion of particles moving along rectilinear paths (only the magnitudes of velocity and acceleration change, not their line of direction).

1. Define position coordinates from fixed datum lines, along the path of each particle. Different datum lines can be used for each particle.

2. Relate the position coordinates to the cord length. Segments of cord that do not change in length during the motion may be left out.

3. If a system contains more than one cord, relate the position of a point on one cord to a point on another cord. Separate equations are written for each cord.

4. Differentiate the position coordinate equation(s) to relate velocities and accelerations. Keep track of signs!
EXAMPLE

Given: In the figure on the left, the cord at A is pulled down with a speed of 2 m/s.

Find: The speed of block B.

Plan:

There are two cords involved in the motion in this example. There will be two position equations (one for each cord). Write these two equations, combine them, and then differentiate them.
Solution:

1) Define the position coordinates from a fixed datum line. Three coordinates must be defined: one for point A \( (s_A) \), one for block B \( (s_B) \), and one for block C \( (s_C) \).

• Define the datum line through the top pulley (which has a fixed position).
• \( s_A \) can be defined to the point A.
• \( s_B \) can be defined to the center of the pulley above B.
• \( s_C \) is defined to the center of pulley C.
• All coordinates are defined as positive down and along the direction of motion of each point/object.
EXAMPLE (continued)

2) Write position/length equations for each cord. Define $l_1$ as the length of the first cord, minus any segments of constant length. Define $l_2$ in a similar manner for the second cord:

\[
\text{Cord 1: } s_A + 2s_C = l_1 \\
\text{Cord 2: } s_B + (s_B - s_C) = l_2
\]

3) Eliminating $s_C$ between the two equations, we get

\[
s_A + 4s_B = l_1 + 2l_2
\]

4) Relate velocities by differentiating this expression. Note that $l_1$ and $l_2$ are constant lengths.

\[
v_A + 4v_B = 0 \quad \Rightarrow \quad v_B = -0.25v_A = -0.25(2) = -0.5 \text{ m/s}
\]

The velocity of block B is 0.5 m/s up (negative $s_B$ direction).
End of the Lecture

Let Learning Continue
12-10 RELATIVE-MOTION ANALYSIS OF TWO PARTICLES USING TRANSLATING AXES

Objectives:
Students will be able to:
1. Understand translating frames of reference.
2. Use translating frames of reference to analyze relative motion.

- Relative Position, Velocity and Acceleration
- Vector & Graphical Methods
When you try to hit a moving object, the position, velocity, and acceleration of the object all have to be accounted for by your mind.

You are smarter than you thought!

Here, the boy on the ground is at \( d = 10 \) ft when the girl in the window throws the ball to him.

If the boy on the ground is running at a constant speed of 4 ft/s, how fast should the ball be thrown?
When fighter jets take off or land on an aircraft carrier, the velocity of the carrier becomes an issue.

If the aircraft carrier is underway with a forward velocity of 50 km/hr and plane A takes off at a horizontal air speed of 200 km/hr (measured by someone on the water), how do we find the velocity of the plane relative to the carrier?

How would you find the same thing for airplane B?

How does the wind impact this sort of situation?
RELATIVE POSITION (Section 12.10)

The absolute position of two particles A and B with respect to the fixed x, y, z reference frame are given by \( \mathbf{r}_A \) and \( \mathbf{r}_B \). The position of B relative to A is represented by

\[
\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A
\]

Therefore, if \( \mathbf{r}_B = (10 \, \mathbf{i} + 2 \, \mathbf{j}) \, \text{m} \)

and \( \mathbf{r}_A = (4 \, \mathbf{i} + 5 \, \mathbf{j}) \, \text{m} \),

then \( \mathbf{r}_{B/A} = (6 \, \mathbf{i} - 3 \, \mathbf{j}) \, \text{m} \).
RELATIVE VELOCITY

To determine the relative velocity of B with respect to A, the time derivative of the relative position equation is taken.

\[ v_{B/A} = v_B - v_A \]

or

\[ v_B = v_A + v_{B/A} \]

In these equations, \( v_B \) and \( v_A \) are called absolute velocities and \( v_{B/A} \) is the relative velocity of B with respect to A.

Note that \( v_{B/A} = -v_{A/B} \).
RELATIVE ACCELERATION

The time derivative of the relative velocity equation yields a similar vector relationship between the absolute and relative accelerations of particles A and B.

These derivatives yield: \( \mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A \)

or

\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \]
SOLVING PROBLEMS

Since the relative motion equations are vector equations, problems involving them may be solved in one of two ways.

For instance, the velocity vectors in $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ could be written as two dimensional (2-D) Cartesian vectors and the resulting 2-D scalar component equations solved for up to two unknowns.

Alternatively, vector problems can be solved “graphically” by use of trigonometry. This approach usually makes use of the law of sines or the law of cosines.

Could a CAD system be used to solve these types of problems?
LAWS OF SINES AND COSINES

Since vector addition or subtraction forms a triangle, sine and cosine laws can be applied to solve for relative or absolute velocities and accelerations. As a review, their formulations are provided below.

Law of Sines: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)

Law of Cosines:
- \( a^2 = b^2 + c^2 - 2bc \cos A \)
- \( b^2 = a^2 + c^2 - 2ac \cos B \)
- \( c^2 = a^2 + b^2 - 2ab \cos C \)
EXAMPLE

Given: \( v_A = 650 \text{ km/h} \quad v_B = 800 \text{ km/h} \)

Find: \( v_{B/A} \)

Plan:

a) Vector Method: Write vectors \( v_A \) and \( v_B \) in Cartesian form, then determine \( v_B - v_A \)

b) Graphical Method: Draw vectors \( v_A \) and \( v_B \) from a common point. Apply the laws of sines and cosines to determine \( v_{B/A} \).
Solution:

a) Vector Method:

\[ v_A = (650 \, \hat{i}) \, \text{km/h} \]
\[ v_B = -800 \cos 60 \, \hat{i} - 800 \sin 60 \, \hat{j} \]
\[ = ( -400 \, \hat{i} - 692.8 \, \hat{j} ) \, \text{km/h} \]

\[ v_{B/A} = v_B - v_A = (-1050 \, \hat{i} - 692.8 \, \hat{j}) \, \text{km/h} \]

\[ v_{B/A} = \sqrt{(-1050)^2 + (-692.8)^2} = 1258 \, \text{km/h} \]

\[ \theta = \tan^{-1}(\frac{692.8}{1050}) = 33.4^\circ \]
EXAMPLE (continued)

b) Graphical Method:

Note that the vector that measures the tip of B relative to A is $v_{B/A}$.

Law of Cosines:

$$(v_{B/A})^2 = (800)^2 + (650)^2 - (800)(650)\cos 120^\circ$$

$v_{B/A} = 1258$ km/h

Law of Sines:

$$\frac{v_{B/A}}{\sin(120^\circ)} = \frac{v_A}{\sin \theta} \quad \text{or} \quad \theta = 33.4^\circ$$
EXAMPLE II

Given: \( v_A = 30 \text{ mi/h} \)
\( v_B = 20 \text{ mi/h} \)
\( a_B = 1200 \text{ mi/h}^2 \)
\( a_A = 0 \text{ mi/h}^2 \)

Find: \( v_{B/A} \)
\( a_{B/A} \)

Plan: Write the velocity and acceleration vectors for A and B and determine \( v_{B/A} \) and \( a_{B/A} \) by using vector equations.

Solution:

The velocity of B is:
\[
v_B = -20 \sin(30) \mathbf{i} + 20 \cos(30) \mathbf{j} = (-10 \mathbf{i} + 17.32 \mathbf{j}) \text{ mi/h}
\]
EXAMPLE II (continued)

The velocity of A is:

\[ \mathbf{v}_A = -30 \mathbf{i} \text{ (mi/h)} \]

The relative velocity of B with respect to A is \( \mathbf{v}_{B/A} \):

\[ \mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A = (-10\mathbf{i} + 17.32\mathbf{j}) - (-30\mathbf{i}) = (20\mathbf{i} + 17.32\mathbf{j}) \text{ mi/h} \]

or

\[ \mathbf{v}_{B/A} = \sqrt{(20)^2 + (17.32)^2} = 26.5 \text{ mi/h} \]

\[ \theta = \tan^{-1} \left( \frac{17.32}{20} \right) = 40.9^\circ \]
EXAMPLE II (continued)

The acceleration of B is:
\[ \mathbf{a}_B = (\mathbf{a}_t)_B + (\mathbf{a}_n)_B = [-1200 \sin(30) \mathbf{i} + 1200 \cos(30) \mathbf{j}] \]

\[ + \left[ \left( \frac{20^2}{0.3} \right) \cos(30) \mathbf{i} + \left( \frac{20^2}{0.3} \right) \sin(30) \mathbf{j} \right] \]

\[ \mathbf{a}_B = 554.7 \mathbf{i} + 1706 \mathbf{j} \text{ (mi/h}^2\text{)} \]

The acceleration of A is zero: \( \mathbf{a}_A = 0 \)

The relative acceleration of B with respect to A is:
\[ \mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A = 554.7 \mathbf{i} + 1706 \mathbf{j} \text{ (mi/h}^2\text{)} \]

\[ \mathbf{a}_{A/B} = \sqrt{(554.7)^2 + (1706)^2} = 1790 \text{ mi/h}^2 \]

\[ \beta = \tan^{-1}(1706 / 554.7) = 72^\circ \]
End of the Lecture

Let Learning Continue